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Interpolation in Fréchet spaces with an application to complex function theory

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ABSTRACT

We give a necessary and sufficient condition on a sequence of linear functionals on a Fréchet space to be interpolating. A practical criterion is given and applied to several natural interpolation problems in complex function theory.

1. INTRODUCTION

Let E be a Fréchet space, E^* the topological dual of E; $L_n \in E^*$, $n=1,2,\dots$ is said to be an interpolating sequence if for any scalar sequence A_n, $n=1,2,\dots$ there exists $f \in E$ such that $L_n(f)=A_n$, $n=1,2,\dots$. The problem is to find conditions on the sequence L_n to ensure that it is an interpolating sequence. Let us note that obviously the answer does not depend on the ordering of the sequence (L_n).

This problem has very natural applications when $E=H(\Omega)$, the space of holomorphic functions in an open set Ω in $\mathbb{C}^n$, endowed with the topology of uniform convergence on compact subsets of Ω.

P. Gauthier and L.A. Rubel have given a necessary and sufficient condition on (L_n) to be an interpolating sequence when E is separable.

DEFINITION 1.1. *For each n, denote by V_n the space span$(L_1,\dots,L_n)$. The sequence (L_n) is said to be totally linearly independent if*

(1) *(L_n) is linearly independent in E^*;*

(2) *If* $T_n \in V_n$, $n \in \mathbb{N}$ *and* $T_n \to 0$ *weakly then there necessarily exists an integer* N *such that* $T_n \in V_N$ *for* $n = 1, 2, \ldots$.

The theorem of Gauthier and Rubel, [4] claims then that if E is separable, in order to be interpolating it is necessary and sufficient that (L_n) is totally linearly independent.

We prove in a different way that the condition remains necessary and sufficient even when E is not separable (Th 2.2). But the main result is theorem 2.3 which gives a useful criterion (see the examples below) to decide if a sequence is interpolating.

In particular, our criterion 2.3 may conveniently be used to extend a theorem of T. Bloom, [3], on a problem of interpolation at discrete subsets of $\mathbb{C}^n$, connected with Kergin polynomial interpolation.

Finally note that there is no interpolating sequence in a Banach space, see [4]; thus throughout the paper Fréchet space means non normable Fréchet space.

2. THE THEOREM ON INTERPOLATION

Let E be a Fréchet space, an increasing sequence $\mathscr{G} = (g_n)$ of semi-norms which defines the topology of E is called an exhaustive sequence.

DEFINITION 2.2. *The index of* $L \in E^*$ *relatively to an exhaustive sequence* $\mathscr{G}$ *is defined to be the least integer* n *such that there exists a finite constant* M *satisfying*

$$|L(f)| \le M g_n(f) \quad (f \in E).$$

We will write $n = \mathit{Ind}\, L$ *and if necessary* $n = \mathit{Ind}_{\mathscr{G}}\, L$.

THEOREM 2.2. *Let* E *be a Fréchet space and* (L_n) *a sequence in* E^* *then* (L_n) *is interpolating if and only if it is a totally linearly independent sequence.*

PROOF. The necessity may be proved exactly in the same way as in [4]. The proof of the sufficiency is done in several steps.

First step. We may find an exhaustive sequence $\mathscr{G}$ such that for each n there exists α_n with $\mathit{Ind}_{\mathscr{G}}\, L_{\alpha_n} = n$.

Indeed, let $\mathscr{G}' = (g_n)$ be any exhaustive sequence. The set $I = \{\mathit{Ind}\, L_i,\ i = 1, 2, \ldots\}$ is unbounded otherwise there exists an integer k_0 such that

$$|L_n(f)| \le M_n g_{k_0}(f) \quad (n = 1, 2, \ldots, f \in E)$$

and this contradicts the fact that (L_n) is totally linearly independent (consider the sequence $T_n = (M_n n)^{-1} L_n$). Now take $\mathscr{G} = \{g_i,\ i \in I\}$.

Second step. We work with the sequence $\mathscr{G}$ defined in the first step. Let W be the space:

$$W = \mathit{span}(L_n,\ n = 1, 2, \ldots)$$

and let W_1 be the subspace of W defined by

$$W_1 = \{L \in W,\ Ind_{\mathscr{G}}(L) \leq 1\}.$$

Since (L_n) is totally linearly independent, W_1 is of finite dimension and by the first step $W_1 \neq \{0\}$. We then choose $v_1^1, \dots, v_{k_1}^1$ a basis of W_1.

Next we consider

$$W_2 = \{L \in W,\ Ind_{\mathscr{G}}(L) \leq 2\}.$$

W_2 is also a finite dimensional subspace of W; by the first step W_2 is strictly larger than W_1 so that we can complete the system $v_1^1, \dots, v_{k_1}^1$ by $v_1^2, \dots, v_{k_2}^2$ to get a basis of W_2. We go on so that for each n considering

$$W_n = \{L \in W,\ Ind_{\mathscr{G}}(l) \leq n\},$$

we construct a finite sequence $v_1^n, \dots, v_{k_n}^n$ and we finally define

$$T = \{v_1^1, \dots, v_{k_1}^1, \dots, v_1^n, \dots, v_{k_n}^n, \dots\}.$$

Note that T is clearly a linearly independent sequence.

Third step. Supposing that T is an interpolating sequence, we show that (L_n) is also an interpolating sequence.

Let A_n be a scalar sequence. We look for $f \in E$ such that $L_n(f) = A_n$ for $n = 1, 2, \dots$. To do this we search a scalar sequence B_n such that the solution of $T_n(f) = B_n$ $n = 1, 2, \dots$ is solution of $L_n(f) = A_n$ $n = 1, 2, \dots$. But for each n we have, for some coefficients $a_n^k, \dots, a_n^{N_n}$, $N_n < \infty$

$$L_n = \sum_{k=1}^{N_n} a_n^k T_k.$$

Hence we only have to prove the existence of a solution for the system

$$A_n = \sum_{k=1}^{N_n} a_n^k B_k.$$

Such a solution exists since the linear forms in the variables $B = (B_1, B_2, \dots)$ defined by $m_n(B) = \sum_{k=1}^{N_n} a_n^k B_k$ are linearly independent (since so are the L_n), see e.g. [7, lemma 6.3.7].

Final step. Now we remark that the sequence T has the following nice property: for each L in the linear span of $v_1^n, \dots, v_{k_n}^n$ we have $L = 0$ or $Ind_{\mathscr{G}}(L) = n$; but it is proved in theorem 2.3 below that such a sequence is an interpolating sequence. So that theorem 2.2 will be proved as soon as theorem 2.3 is. □

COROLLARY (to the proof). *Let E be a Fréchet space, $\mathscr{G}$ an exhaustive sequence of E and (L_n) a linearly independent sequence in E^*. Then (L_n) is interpolating if and only if there exists a linearly independent sequence (T_n) such that*

(1) *Each L_n is a linear combination of some T_n's.*

(2) *There exists a sequence of integers $1 = n_1 < n_2 < n_3 < \cdots$ such that if*

T is in the linear span of the T_k for $k=n_i, n_i+1, \dots, n_{i+1}-1$ then $L=0$ or $Ind_{\mathscr{G}}(L)=i$.

THEOREM 2.3. *Let E be a Fréchet space, $\mathscr{G}$ an exhaustive sequence and (L_n) a linearly independent sequence in E^*. Then if condition $(*)$ below holds, (L_n) is an interpolating sequence.*

$(*)$ There exists a sequence of integers n_i, $i=1,2,\dots$; $1=n_1<n_2<\cdots$ such that for each $L\in sp\{L_{n_i}, L_{n_i+1}, \dots, L_{n_{i+1}-1}\}$, $L\neq 0$ implies $Ind_{\mathscr{G}}(L)=i$.

In applications (see below) there is often a *natural* and *convenient* choice for the exhaustive sequence $\mathscr{G}$ and the ordering of L_n for which $(*)$ can be proved without too many difficulties. Here lies the interest of the criterion.

We first recall some facts about Fréchet spaces, see [12, 5.4] for details.

Let $\mathscr{G}$ be an exhaustive sequence of E. Then each space $E/g_n^{-1}(0)$ is normed by $|\tilde{x}|=g_n(x)$ if $x\in\tilde{x}$. Let E_n be the completion of this normed space; the natural restriction map from $E/g_n^{-1}(0)$ to $E/g_m^{-1}(0)$ when $m\leq n$ extends to a continuous linear map $p_{mn}\in\mathscr{L}(E_n, E_m)$ $(m\leq n)$ whose norm is less than or equal to 1 and then E is the projective limit of the family $\{p_{mn}E_n,\ n\in\mathbb{N}\}$. If we let x_n be the class of x in $E/g_n^{-1}(0)$, for $m<n$ and $x\in E$ we always have: $x_m=p_{mn}(x_n)$. In the sequel the same notation $(|\cdot|)$ is used for all the different norms.

In the following two lemmas we work with the sequence $\mathscr{G}$ of theorem 2.3.

LEMMA 2.4. *Let $L_i\in E^*$, $i=1,\dots,d$ and $\alpha\in\mathbb{N}^*$.*

If for each $L\in span\{L_1,\dots,L_d\}$, $L\neq 0$ implies $Ind_{\mathscr{G}}(L)>\alpha$, then the following approximation property holds:

For each $h\in E$, $\varepsilon>0$ there exists $a=a(h,\varepsilon)\in E$ such that

(1) $|(h-a)_\alpha|\leq\varepsilon$;

(2) $L_i(a)=0$, $i=1,\dots,d$.

PROOF. Let

$$N=\bigcap_{i=1}^{d} Ker\, L_i,$$

$N_\alpha=\{x_\alpha,\ x\in N\}$, μ a continuous linear form on E_α, orthogonal to N_α (that is, $\mu(y)=0$ for $y\in N_\alpha$).

Define, for $x\in E$, $\bar{\mu}(x)=\mu(x_\alpha)$. Then $\bar{\mu}$ is a continuous linear form on E whose index is less than or equal to α. But $\bar{\mu}$ is null on N which implies by a well known lemma that

$$\bar{\mu}=\sum_{i=1}^{d}\lambda_i L_i.$$

By hypothesis the only possibility is $\bar{\mu}=0$ and then also $\mu=0$ so that by the Hahn-Banach theorem N_α is dense in E_α and the lemma is proved. □

LEMMA 2.5. *Let $d\leq k$, $L_i\in E^*$ for $i=1,\dots,k$, $A_j\in\Delta$, $j=1,\dots,d$ (Δ is the field* *of scalars) and $f \in E$. Suppose that*

(1) *the L_i, $i=1,\dots,d$ are linearly independent in E^*,*

(2) *For each $L \in span\{L_1,\dots,L_d\}$, $L \neq 0$, we have $Ind\, L > \alpha$,*

(3) *For $j=d+1,\dots,k$, $Ind\, L_j \leq \alpha$.*

Then given $\varepsilon>0$ there exists $h \in E$ such that

(4) $|h_\alpha - f_\alpha| \leq \varepsilon$;

(5) $L_i(h) = A_i$, $i=1,\dots,d$;

(6) $L_j(h) = L_j(f)$, $j=d+1,\dots,k$.

PROOF. We may suppose, taking away if necessary some of the L_j for $j=d+1,\dots,k$ that the L_i, $i=1,\dots,k$ are linearly independent in E^*. Then there exist vectors l_j such that

$$L_i(l_j) = \delta_{ij}, \quad i,j=1,\dots,k.$$

First consider the linear map

$$\mathscr{K} : E \ni x \to \mathscr{K}(x) = \sum_{i=d+1}^{k} L_i(x)\, l_i \in E.$$

Obviously $\mathscr{K}$ is continuous and by hypothesis (3) it satisfies

$$(\dagger) \qquad |(\mathscr{K}x)_\alpha| \leq C|x_\alpha|, \quad x \in E$$

where C is some finite constant.

Now define

$$r = f - \sum_{j=1}^{d} A_j l_j.$$

By lemma 2.4 there exists $a^n \in E$ such that a^n_α converges to r_α as n tends to ∞ and $L_i(a^n)=0$ for $i=1,\dots,d$, $n \in \mathbb{N}^*$.

Finally for each n define

$$r^n = a^n - \mathscr{K}(a^n - r) + \sum_{j=1}^{d} A_j l_j.$$

r^n satisfies conditions (5) and (6) of the lemma. Moreover for n large enough r^n satisfies condition (4) since a^n_α converges to r_α and $(\mathscr{K}(a^n - r))_\alpha$ tends to 0 when n approaches ∞ because of the inequality ($\dagger$). □

PROOF OF 2.3. We are going to construct a sequence x^n in E by induction. First we choose $x^1 \in E$ such that for $i=1,\dots,n_2-1$

$$L_i(x^1) = A_i.$$

Supposing that x^k has been chosen, we choose x^{k+1} with the following properties:

(i) $L_i(x^{k+1})=0$, $i=1,2,\dots,n_{k+1}-1$;

(ii) $L_i(x^{k+1}) = A_i - \sum_{j=1}^{k} L_i(x^j)$ for $i=n_{k+1}, n_{k+1}+1,\dots,n_{k+2}-1$;

(iii) $|(x^{k+1})_k| \leq 1/2^k$.

Such a choice is possible, taking account of the condition $(*)$ of theorem 2.3, and then applying lemma 2.5 to the origin 0 of E.

Now for each n the series

$$\sum_{d\geq 0} x_n^d$$

is normally convergent in E_n, hence the series converges (here we use the completeness of E_n or equivalently of E). Hence the series

$$\sum_{d\geq 0} x^d$$

converges in E and its sum f satisfies $L_i(f)=A_i$ for each $i=1,2,\ldots$. Thus the theorem is proved. □

3. EXAMPLES

We give just a few examples. Some references are given in a note below; nevertheless an absence of reference does not mean the result is new.

EXAMPLE 3.1. *Let Ω be a domain of holomorphy in $\mathbb{C}^m$, (z_i) a discrete sequence in Ω and $p_i(z)$ a sequence of polynomials with degree of $p_i(z)$ equal to $d(z_i)=d_i$. Then there exists $f\in H(\Omega)$ such that $p_i(z)$ is the $d(z_i)^{th}$ Taylor polynomial of f at the point z_i.*

PROOF. We want to apply theorem 2.3 with $E=H(\Omega)$ and the continuous linear functionals

$$D^\beta_{z_i}: f\to (D^\beta f)(z_i),\ |\beta|\leq d(z_i),\quad i=1,2,\ldots$$

where as usual $\beta=(\beta_1,\ldots,\beta_m)$, $|\beta|=\sum\beta_i$ and $D^\beta=\partial^{|\beta|}/\partial^{\beta_1}z_1\cdots\partial^{\beta_m}z_m$.

Choose (K_n) an exhaustive sequence of $H(\Omega)$-convex compact subsets of Ω such that none of the z_i lie on the boundary of some K_n and reorder the sequence (z_i) in such a manner that $z_{n_j},\ldots,z_{n_{j+1}-1}\in \mathring{K}_j\setminus K_{j-1}$. We are going to show that condition $(*)$ of 2.3 holds with the exhaustive sequence $\mathscr{G}$ defined by $g_n=|\cdot|_{K_n}$, that is, the sup norm on the compact subset K_n.

Let $L\neq 0$, L belonging to the linear span of the D^β_z, $|\beta|\leq d(z)$ for $z\in\{z_{n_j},\ldots,z_{n_{j+1}-1}\}$. Clearly we have $Ind\,L\leq j$. Suppose that $Ind\,L\leq j-1$. Then by the Hahn-Banach theorem L extends continuously to $\mathscr{C}(K_{j-1})$. Since K_{j-1} is $H(\Omega)$-convex and $z_{n_j}\notin K_{j-1}$ we may find $f_{n_j}\in H(\Omega)$ such that $|f_{n_j}(z_{n_j})|>|f|_{K_{j-1}}$; similarly we take $f_{n_j+1},\ldots,f_{n_{j+1}-1}$ and finally $f\in H(\Omega)$ such that $L(f)\neq 0$. Then the function $F(z)$ defined by

$$f(z)/[(f_{n_j}(z)-f_{n_j}(z_{n_j}))^{d_{n_j}+1}\cdots(f_{n_{j+1}-1}(z)-f_{n_{j+1}-1}(z_{n_{j+1}-1}))^{d_{(n_{j+1}-1)}+1}]$$

is analytic in a neighborhood of K_{j-1}. Hence, see [7, cor 5.2.9], F is the uniform limit on K_{j-1} of a sequence $h_n\in H(\Omega)$. Thus f is the uniform limit on K_{j-1} of h_nG where G is the denominator in the above formula, but in this case $0\neq L(f)=lim_{n\to\infty}L(h_nG)=0$. Contradiction! We then conclude that $Ind\,L=j$ and according to 2.3 the claim is proved. □

EXAMPLE 3.2. *Let (μ_i) be a sequence of analytic functionals in $\mathbb{C}^n$. Suppose that each μ_i admits a convex (resp. polynomially convex, resp. holomorphically convex) support C_i such that C_i is relatively compact in the interior of C_{i+1} and $\bigcup C_i = \mathbb{C}^n$. Then (μ_i) is an interpolating sequence for $H(\mathbb{C}^n)$.*

This follows immediately from 2.3 and the definition of a support, see [6].

EXAMPLE 3.2′. *Let ϱ be a norm on $\mathbb{C}^n$ and (μ_i) a sequence of analytic functionals, and let σ_i be the ϱ-type of the Fourier-Borel transform of μ_i. Then if $\sigma_i \neq \sigma_j$ for $i \neq j$, (μ_i) is an interpolating sequence for $H(\{\varrho^*(z) \leq \sup \sigma_i\})$.*

Indeed, by a theorem of Martineau, see [9, page 76], the above conditions mean that $\{\varrho^*(z) \leq \sigma_i\}$ is a ϱ^*-support of μ_i, ϱ^* being the dual norm of ϱ.

EXAMPLE 3.3. *Let Ω be an open subset of $\mathbb{R}^s$, (a_n) a discrete sequence in Ω. Then D^α at (a_n), $|\alpha| = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$ is an interpolating (double) sequence for $C^\infty(\Omega)$.*

EXAMPLE 3.4. *Let A^∞ be the space of analytic functions F such that $F, F, F'', \ldots$ are all uniformly continuous for $|z| < 1$. Then the sequence $f \to f^{(n)}(1)$, $n = 0, 1, 2, \ldots$ is an interpolating sequence for A^∞.*

Indeed, A^∞ is a Fréchet space with the (convenient) system of semi-norms: $p_n(f) = \sup\{|f^{(k)}(e^{i\theta})|,\ 0 \leq \theta \leq 2\pi,\ 0 \leq k \leq n\}$, $n = 0, 1, 2, \ldots$.

NOTE 3.5. There are many proofs of 3.1 in case $n = 1$; see [8] and the references there. In particular this result is proved in the case where Ω is a ball and $d(z_i) = 0$ by Walsh, see [14, chap 11], using polynomial approximation with auxiliary conditions. It is Walsh's viewpoint which is developed in theorem 2.3. In case $n > 1$, 3.1 follows also for example from Cartan's extension theorem. 3.3 is a well known theorem of Borel. Example 3.4 may be found in [13, page 455]; Taylor proves it using a non trivial isomorphism $A^\infty \sim E^*$ where E is some space of entire functions.

4. INTERPOLATION AT DISCRETE SUBSETS OF $\mathbb{C}^n$

We first recall some facts about Kergin polynomial interpolation.

Let $\mathscr{A} = (A_0, \ldots, A_d)$ be not necessarily distinct points in $\mathbb{C}^n$ and let Ω be an open neighborhood of the convex hull of $\mathscr{A}$.

Then there exists a unique continuous linear projector

$$\mathscr{K}_{\mathscr{A}} : H(\Omega) \to \mathscr{P}_d(\mathbb{C})$$

where $\mathscr{P}_d(\mathbb{C})$ is the space of complex polynomials of degree $\leq d$ on $\mathbb{C}^n$ such that

(i) $\mathscr{K}_{\mathscr{A}}(f)(A_i) = f(A_i)$, $i = 0, 1, \ldots, d$;

(ii) If $f = g \circ \lambda$ where λ is an affine map from $\mathbb{C}^n$ to $\mathbb{C}^m$ and g analytic in $\lambda(\Omega)$, then

$$\mathcal{K}_{\mathcal{A}}(f) = \mathcal{K}_{\lambda(\mathcal{A})}(g) \circ \lambda,$$

where $\lambda(\mathcal{A}) = \{\lambda(A_0), \lambda(A_1), \ldots, \lambda(A_d)\}$;

(iii) the map $\mathcal{K}_{\mathcal{A}}$ does not depend on the ordering of the sequence A;

(iv) If $\mathcal{B} \subset \mathcal{A}$ then $\mathcal{K}_{\mathcal{B}} \circ \mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{B}}$.

Property (ii) is of particular interest when $m = 1$ since in this case $\mathcal{K}_{\mathcal{A}}(g)$ is the classical Lagrange-Hermite polynomial of g at the points $\lambda(A_0), \lambda(A_1), \ldots, \lambda(A_d)$.

The Kergin interpolant may be written in the following form, called Newton form,

$$\mathcal{K}_{\mathcal{A}}(f) = \sum_{|\alpha \leq d} b_\alpha(f) Q_\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \sum_{k=1}^n \alpha_k$, $(Q_\alpha, |\alpha| \leq d)$ forms a basis of $\mathcal{P}_d(\mathbb{C})$ and the "b_α" are analytic functionals defined by

$$b_\alpha(f) = \int_{\Delta^i} D^\alpha(f)(\sum_{j=1}^{i} A_0 + \lambda_j(A_j - A_0))\, d\lambda$$

if $i = |\alpha|$ and where $\Delta^i = \{(\lambda_j)_{j=1,\ldots,i}, 1 \leq j \leq 1, \sum_{j=1}^i \lambda_j \leq 1\}$.

The functional b_α and the basis polynomials Q_α are linked by the orthogonality property:

$$b_\alpha(Q_\beta) = \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta}$ is the usual Kronecker symbol.

In case $n = 1$, $Q_i(z)$ is the i^{th} Newton basis polynomial, that is

$$Q_i(z) = (z - A_0) \cdots (z - A_{i-1}),$$

and $b_i(f)$ is the i^{th} divided difference of f with respect to the points $A_0, \ldots, A_i$. Similarly in the several variable case the basis polynomial Q_α depends only on $A_0, \ldots, A_{i-1}$ if $|\alpha| = i$. Details and further information about Kergin interpolation may be found e.g. in [1], [2], [10], [11].

The following interpolation theorem is proved by Bloom, [3] when $\Omega = \mathbb{C}^n$ in a different (more technical) manner.

THEOREM 4.1. *Let $A_0, A_1, \ldots$ be a discrete sequence in an open convex subset Ω of $\mathbb{C}^n$ and $\sum_{\alpha \in \mathbb{N}^n} a_\alpha Q_\alpha(z)$ a formal Newton series. Then there exists $f \in H(\Omega)$ such that*

$$\mathcal{K}_d(f) = \sum_{|\alpha| \leq d} a_\alpha Q_\alpha, \quad d = 0, 1, \ldots$$

where $\mathcal{K}_d(f)$ is the Kergin interpolant of f at the points $A_0, A_1, \ldots, A_d$.

REMARK 4.2. In the one variable case the theorem 4.1 says exactly the same as 3.1, see [3, page 1224].

PROOF OF 4.1. As Bloom noticed, [3, 4.3] the problem does not depend on the ordering of the sequence, this comes from (iii) and (iv). So by changing the numbering we order the sequence as:

$$\underbrace{A_0,\dots,A_0}_{m_0}\ \underbrace{A_1,\dots,A_1}_{m_1}\ \underbrace{A_2,\dots,A_2}_{m_2}\dots\dots$$

and in such a manner that A_i does not lie in the convex hull of $\{A_0, A_1, \dots, A_{i-1}\}$. The above notation means that A_0 appears m_0 times in the sequence, A_1 appears m_1 times, etc.

For simplicity we define $n_0 = 1$, $n_1 = m_0 + 1$, $n_2 = m_0 + m_1 + 1$, etc., hence if $|\alpha| = d$ and $d = n_k + j$ with $0 \le j < m_k$ then

$$b_\alpha(f) = \int_{\Delta^d} (D^\alpha f)(A_0 + \sum_{i=n_1}^{n_2-1} \lambda_i(A_1 - A_0) + \dots + \sum_{i=n_k}^{n_k+j} \lambda_i(A_k - A_0))\, d\lambda.$$

The theorem will be proved if we find $f \in H(\Omega)$ such that:

$$b_\alpha(f) = a_\alpha, \quad |\alpha| = 0, 1, 2, \dots.$$

In other words, the question is:

Is the sequence b_α, $|\alpha| = 0, 1, 2, \dots$ an interpolating sequence for $H(\Omega)$?

Let C_n, $n = 0, 1, 2, \dots$ be an exhaustive sequence of compact convex subsets of Ω such that $A_i \in \mathring{C}_i \setminus C_{i-i}$. We consider the exhaustive sequence $\mathscr{G}$ defined by $|\cdot|_{C_n}$, $n = 0, 1, 2, \dots$ and want to use theorem 2.3 with it. More precisely we want to show that if L is in the linear span of the b_α for $n_k \le |\alpha| < n_{k+1}$, then $Ind_{\mathscr{G}}\, L = k$ or $L = 0$.

So let L be such a functional, that is

$$L = \sum_{n_k \le |\alpha| < n_{k+1}} l_\alpha b_\alpha, \quad l_\alpha \in \mathbb{C}. \tag{1}$$

Since $A_0, A_1, \dots, A_k$ lie in $\mathring{C}_k$ we have $Ind_{\mathscr{G}}\, L \le k$ (note the fact that each b_α appearing in L has A_k in its definition).

Supposing that $Ind_{\mathscr{G}}\, L \le k-1$, we are going to show that this leads to $L = 0$.

Defining $F_a(z) = \langle A_k - z, a\rangle$, there exists an open set of points $a \in \mathbb{C}^n$ such that $F_a(C_{k-1})$ does not contain $F_a(A_k) = 0$. Further if the functional $F_a * L$ is defined by

$$(F_a * L)(g) = L(g \circ F_a), \quad g \in H(F_a(\Omega)),$$

then by (1) and using similar notation we have:

$$F_a * L = \sum_{n_k \le |\alpha| < n_{k+1}} l_\alpha (F_a * b_\alpha). \tag{2}$$

However, cf. [10] or [11]:

$$(F_a * b_\alpha)(g) = a^\alpha [F_a(A_0), \dots, F_a(A_k)](g) \tag{3}$$

where the term in brackets denotes the divided difference of g at the points $F_a(A_0), \dots, F_a(A_k) = 0$. Hence the functional (2) is discrete since it is a sum of divided differences; moreover if $Ind_{\mathscr{G}}\, L \le k-1$ this functional is carried by

$F_a(C_{k-1})$ which does not contain $F_a(A_k)(=0)$. We conclude that the contribution of 0 in (2) must be zero. Looking at classical formulas for divided differences, see [5, page 35], we find that this contribution is:

$$(4) \qquad \sum_{j=0}^{m_k-1} \sum_{|\alpha|=n_k+j} l_\alpha a^\alpha \sum_{m=0}^{j+1} \frac{g^{j-m}(0)}{m!(j-m)!} \frac{d^m}{du^m}\left(\frac{u^{j+1}}{Q(u)}\right)\Bigg|_{u=0}$$

where $Q(u)=(u-F_a(A_0)(u-F_a(A_1))\cdots(u-F_a(A_k))$. If we look at the coefficient of the term $g^{m_k-1}(0)$ in (4) we get that

$$\sum_{|\alpha|=n_k+m_k-1} l_\alpha a^\alpha = 0;$$

since this is true for an open set of points a we necessarily have $l_\alpha=0$ for $|\alpha|=n_k+m_k-1$. Hence we may take away the index $j=m_k-1$ in (4). Afterwards we proceed similarly for the coefficient of the term $g^{m_k-2}(0)$ and we conclude that $l_\alpha=0$ for $|\alpha|=n_k+m_k-2$. So we go on and finally we obtain $L=0$. The theorem is proved. □

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